Countably Coverable Rings

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Motivation: Covering Numbers of Groups

Let G be a group.

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Easy Observations and General Questions

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- G is coverable if and only if G is non-cyclic.
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Let R be a ring.

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  - Does there exist $n \in \mathbb{N}$ such that no ring has covering number n?

Coverable Rings, Singly Generated Rings

Definition

A ring R is *singly generated* if it can be generated (as a ring) by a single element.

Examples/Non-examples:

- $\mathbb{Z}$ is singly generated (1 is a generator)
- Any finite field is singly generated (generator of unit group generates the entire field)
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Lemma (O.-W. 2022)

Let D be a commutative integral domain (possibly without unity).
Then, D is singly generated if and only if one of the following holds:

1. D is a finite field
2. $D \cong x\mathbb{F}_p[x] = \{xf(x) : f(x) \in \mathbb{F}_p[x]\}$ for some prime p
3. $D \cong \alpha\mathbb{Z}[\alpha] = \{\alpha f(\alpha) : f(x) \in \mathbb{Z}[x]\}$ for some $\alpha \in \mathbb{C}$

Rings with a Finite Cover

Let R be a coverable ring such that $\sigma(R)$ is finite.

- (E. Swartz-W. 2021+): If R does not have unity, then there exists a unital ring R' such that $\sigma(R) = \sigma(R')$. Thus, it suffices to consider rings with unity.
- (B. H. Neumann (1954), J. Lewin (1967)): There exists a two-sided ideal I of R such that $\sigma(R) = \sigma(R/I)$. Thus, it suffices to consider finite rings with unity.

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Theorem (E. Swartz-W. 2021+)

1. Assume that R is ring with unity and $\sigma(R)$ is finite.
   Then, R has a residue ring R/I such that
   - $\sigma(R) = \sigma(R/I)$
   - R/I is finite of characteristic p
   - R/I falls into one of four (infinite) families, depending on whether it is commutative/noncommutative, semisimple/not semisimple
2. Almost all positive integers are **not** the covering number of a ring.

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Back up a minute: What about groups?

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Example (O.-W. 2022)

Let G be Shelah's group of cardinality $\aleph_1$.
Then, the group algebra $\mathbb{Z}[G]$ is coverable, but not countably coverable.

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- For each $i \in \mathbb{Z}^+$, let $F_i \subsetneqq \mathbb{C}$ be the algebraic closure of $\mathbb{Q}(S \setminus \{x_i\})$.

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- For each $i \in \mathbb{Z}^+$, let $F_i \subsetneq \mathbb{C}$ be the algebraic closure of $\mathbb{Q}(S \setminus \{x_i\})$.
- Show that $\mathbb{C} = \bigcup_{i=1}^{\infty} F_i$ and $\mathbb{R} = \bigcup_{i=1}^{\infty} (F_i \cap \mathbb{R})$.

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- Show that $\mathbb{C} = \bigcup_{i=1}^{\infty} F_i$ and $\mathbb{R} = \bigcup_{i=1}^{\infty} (F_i \cap \mathbb{R})$.
- Thus, both $\mathbb{C}$ and $\mathbb{R}$ are countably coverable.

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If R is Noetherian and not singly generated, then R is countably coverable.

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- Lifting covers from R/P to R shows that R is countably coverable. □

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THANK YOU!