Recasting the Hazrat Conjecture:

Graded Morita equivalence of Leavitt path algebras

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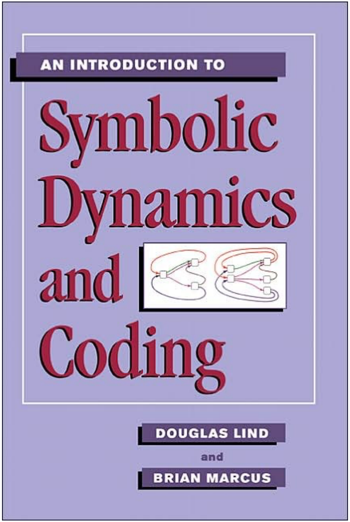
This is joint work with

Efren Ruiz and Mark Tomforde

Standard reference; Cambridge U. Press, 1995



Bi-infinite strings, shift spaces

The original motivating idea: dynamical systems

(e.g. motion of planets, molecules in a gas, ...)

“Discrete time” representation.

i^{th} coordinate is the state of the system at time i.

Original example: Hadamard (1898) applied the idea to geodesic flows on surfaces of negative curvature.

Now: data storage and transmission.

Bi-infinite strings, shift spaces

"Full shift space" on an alphabet $\mathcal{A}$.

$\{(a_i)_{i\in\mathbb{Z}} \mid a_i \in \mathcal{A}\}$. Denote by $\mathcal{A}^{\mathbb{Z}}$.

Elements of $\mathcal{A}^{\mathbb{Z}}$ are called "points".

For $N \in \mathbb{Z}^{\geq 1}$, elements of $\mathcal{A}^N$ are called "blocks of length N".

"shift map" $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$

$$\sigma(x) = y = (y_i) \quad \text{has} \quad y_i = x_{i+1}.$$

Bi-infinite strings, shift spaces

Intuition: a function φ from $\mathcal{A}^{\mathbb{Z}}$ to $\mathcal{A}^{\mathbb{Z}}$ is “time-independent” if the operation acts the same in each coordinate of $\mathcal{A}^{\mathbb{Z}}$.

Example: $\varphi : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$,

$$\varphi((x_i)) = (y_i), \quad \text{where } y_i = x_i + x_{i+1} \ mod 2.$$

Formally: Same as: $\sigma \circ \varphi = \varphi \circ \sigma$.

That is, φ commutes with the shift map.

Bi-infinite strings, shift spaces

Let $\mathcal{F}$ denote a set of blocks from alphabet $\mathcal{A}$.

(a finite or infinite set; blocks need not be of the same size.)

Notation: $\mathbf{X}_{\mathcal{F}}$ denotes the subset of $\mathcal{A}^{\mathbb{Z}}$ consisting of those points which do not contain any blocks in $\mathcal{F}$ (anywhere ...)

$\mathcal{F}$ for "forbidden".

Example: $\mathcal{A} = \{0, 1\}, \quad \mathcal{F} = \{(11)\}$.

Example: $\mathcal{A} = \{0, 1\}, \quad \mathcal{F} = \{(10^{2n+1}1) \mid n \in \mathbb{Z}^{\geq 0}\}$.

Definition: A "shift space on $\mathcal{A}$" is a subset of $\mathcal{A}^{\mathbb{Z}}$ of the form $\mathbf{X}_{\mathcal{F}}$ for some set of forbidden blocks $\mathcal{F}$.

Bi-infinite strings, shift spaces

Note: If $X = \mathbf{X}_{\mathcal{F}}$ for some set of blocks $\mathcal{F}$ built from $\mathcal{A}$, then $\sigma(X) \subseteq X$. So "shift space" is not unreasonable notation.

(BUT, there are subsets of $\mathcal{A}^{\mathbb{Z}}$ which are closed under σ which are not shift spaces; e.g. the set of points in $\{0,1\}^{\mathbb{Z}}$ containing exactly one 1.)

Remark: In Hadamard's original work, the set of points of interest turned out to be a shift space.

Bi-infinite strings, shift spaces

Definition: Two shift spaces X and Y are *conjugate* in case there is a bijection $\varphi : X \to Y$ that

1. commutes with the shift operator σ, and
2. is "locally computable". (not today ...)

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Intuition: Conjugate shift spaces share many common properties.

Note: there is a "natural" topology on shift spaces; in this setting, "conjugate" is "topologically isomorphic".

Shift spaces of finite type

Definition: $\mathcal{A}$ a finite alphabet. A

"shift space of finite type over $\mathcal{A}$"

is a subset X of $\mathcal{A}^{\mathbb{Z}}$ of the form $X = \mathbf{X}_{\mathcal{F}}$ for some finite set $\mathcal{F}$ of blocks taken from $\mathcal{A}$.

Usually just call this a "shift of finite type".

Theorem: If X is a shift of finite type, and X is conjugate to Y, then Y is a shift of finite type.

(proof is nontrivial)

Shift spaces of finite type arising from directed graphs

The Key Example. Let E be a finite (directed) graph.

Vertex set E^0, edge set E^1.

Define the "bi-infinite paths" in E.

$$X = X_E = \{(e_i)_{i\in\mathbb{Z}} \mid e_i \in E^1, r(e_i) = s(e_{i+1}) \ \forall i \in \mathbb{Z}\}.$$

Then X_E is a shift space. "Edge shift space," or just "edge shift".

Actually X_E is a shift space of finite type.

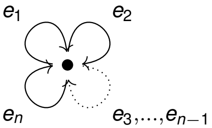
$$X_E = \mathbf{X}_{\mathcal{F}},$$

where $\mathcal{F}$ is the set of all blocks of edges (e, f) of length 2 for which $r(e) \neq s(f)$.

Shift spaces of finite type arising from directed graphs

Examples.

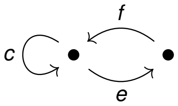
1. Let R_n be the "rose with n petals" graph



Then X_{R_n} is the full shift space on the set $\mathcal{A} = \{e_1, e_2, \ldots, e_n\}$.

Shift spaces of finite type arising from directed graphs

2. Let E be this graph:



Here are some elements of X_E:

$(..., c, c, c, c, c, ...)$

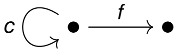
$(..., c, e, f, e, f, c, c, c, e, f, c, c, ...)$

Any shift of the previous one ...

For this graph, $X_E = \mathbf{X}_{\mathcal{F}}$, where $\mathcal{F} = \{(e, e), (f, f), (c, f), (e, c)\}$.

Shift spaces of finite type arising from directed graphs

3. Let T denote the Toeplitz graph



Here are some elements of X_T:

$(\ldots, c, c, c, c, c, \ldots.)$

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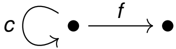

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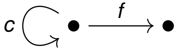

Here are some elements of X_T:

$(..., c, c, c, c, c, ...)$

Actually that's the ONLY element of X_T.

At least as far as shift spaces go, for a graph E, any edges that either go into a sink, or emit from a source, cannot be part of a point in X_E. So from a shift space point of view,

It is usually assumed that E has neither sinks nor sources.

"Essential" graph.

Shift spaces of finite type arising from directed graphs

Remark: not every shift space of finite type arises as the edge shift space of a directed graph. (Easy examples exist.) But:

Theorem (2.3.2, LM) : Every shift space of finite type is conjugate to an edge shift space.

Shift spaces of finite type arising from directed graphs

The information contained in a finite directed graph E can be encoded in its adjacency matrix $A = A_E$.

Examples above.

Notation: the edge shift space X_E is often denote X_A when $A = A_E$.

Shift spaces of finite type arising from directed graphs

Big question: given essential graphs E and F, when is X_E conjugate to X_F?

There are some "basic", easy-to-describe operations that can be performed on graphs.

"Outsplitting", "Insplitting",

"Outamagamation", "Inamalgamation".

Proposition: If G_2 is produced from G_1 by one of these operations, then X_{G_1} and X_{G_2} are conjugate.

Proof is not too hard.

Shift spaces of finite type arising from directed graphs

Intriguing / nice:

Shift spaces of finite type arising from directed graphs

Intriguing / nice:

Theorem: (Decomposition Theorem) Let E and F be finite essential graphs. Then every conjugacy $\varphi : X_E \to X_F$ is the composition of a sequence of conjugacies arising from the four basic operations.

Proof is somewhat technical but does not require any heavy machinery.

Conjugacy of edge shift spaces: the matrix viewpoint

Can the Decomposition Theorem be cast in matrix terms?

That is, can we give relationships between the adjacency matrices A_E and A_F that are equivalent to the condition that the edge shift spaces X_E and X_F are conjugate?

Conjugacy of edge shift spaces: the matrix viewpoint

Definition: C, D square matrices (not necessarily of the same size) with entries in $\mathbb{Z}^{\geq 0}$. C and D are

elementary strong shift equivalent

in case there exist rectangular matrices R, S with entries in $\mathbb{Z}^{\geq 0}$ for which $C = RS$ and $D = SR$.

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Definition: A, B square matrices (not necessarily of the same size) with entries in $\mathbb{Z}^{\geq 0}$. A and B are

strong shift equivalent

if there exists a sequence of square matrices $A = A_1,\ A_2,\ \ldots,\ A_\ell = B$ for which A_i is elementary strong shift equivalent to A_{i+1} for all $1 \leq i \leq \ell - 1$.

Conjugacy of edge shift spaces: the matrix viewpoint

Note: easily “Elementary strong shift equivalence” is reflexive and symmetric. But NOT transitive.

But “Strong shift equivalence” is an equivalence relation on the set of square matrices (of any finite size) with entries in $\mathbb{Z}^{\geq 0}$.

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Note: if $C = RS$ and $D = SR$ then

$$CR = (RS)R = R(SR) = RD. \text{ Similarly } SC = DS.$$

Conjugacy of edge shift spaces: the matrix viewpoint

Theorem (Williams, 1973). Let E and F be finite essential graphs. Then the edge shift spaces

$$X_E \text{ and } X_F \quad \text{are conjugate}$$

if and only if the adjacency matrices

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The proof relies heavily on the Decomposition Theorem, together with a matrix description of the relation between graphs that are constructed using the basic graph moves.

More about matrices

Definition. Two square matrices A and B (not necessarily of the same size) with entries in $\mathbb{Z}^{\geq 0}$ are called

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provided that there exists a positive integer n, and rectangular matrices R and S with entries in $\mathbb{Z}^{\geq 0}$, for which

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Using previous observation:
The $n = 1$ case of shift equivalence is the same as elementary strong shift equivalence.

More about matrices

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Lemma: If A and B are strong shift equivalent, then they are shift equivalent. In fact, the length ℓ of the strong shift equivalent chain can be used as the exponent on the matrices A and B in the shift equivalence definition.

Idea of proof: ($\ell = 2$ case)

Suppose $A = RS, B = SR, \;\; B = TU, C = UT$.

Then easily $A^2 = (RT)(US), \; C^2 = (US)(RT)$,

and the other two equations hold easily.

More about matrices

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The smallest currently-known counterexamples are pairs of 7×7 matrices.

The proof that the two matrices are not strong shift equivalent requires building a deep new invariant (the "sign gyration homomorphism").

More about matrices

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But there is something in between (SE) and (SSE) ...

Module Shift Equivalence

Definition: Let k be a field. For a set S,

$$kS := k\text{-vector space with basis } S; \quad \text{i.e., } \bigoplus_{s \in S} k.$$

Let $E = (E^0, E^1, r, s)$ be a (finite) directed graph.

kE^0 is not only a vector space, but a ring in the expected way:

$kE^0 \cong \bigoplus_{v \in E^0} k$

kE^1 has no 'natural' algebra structure.

But kE^1 is a $kE^0 - kE^0$-bimodule in the expected way.

Module Shift Equivalence

Definition: The finite graphs E and F are called

Module Shift Equivalent

(with exponent n) in case there exists a kE^0–kF^0-bimodule M, a kF^0–kE^0-bimodule N, and a positive integer n for which there are bimodule isomorphisms

$$(kE^1)^{\otimes n} \cong M \otimes_{kF^0} N \qquad (kF^1)^{\otimes n} \cong N \otimes_{kE^0} M$$

$$kE^1 \otimes_{kE^0} M \cong M \otimes_{kF^0} kF^1 \qquad kF^1 \otimes_{kF^0} N \cong N \otimes_{kE^0} kE^1.$$

Module Shift Equivalence

Theorem: [Shift Equivalence $\Leftrightarrow$ Module Shift Equivalence] Let k be a field, and let E and F be finite graphs with no sinks. Then these are equivalent:

1. A_E and A_F are shift equivalent (with exponent n).
2. E and F are Module Shift Equivalent (with exponent n).

Proof: Not bad. (Uses properties of "polymorphisms".)

Condition (Com)

Now suppose there exists some collection of such bimodule isomorphisms that behaves nicely.

Condition (Com)

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Specifically, suppose $\exists$ M, N, n, and bimodule isomorphisms

$$\omega_E : M \otimes_{kF^0} N \to (kE^1)^{\otimes n} \qquad \omega_F : N \otimes_{kE^0} M \to (kF^1)^{\otimes n}$$

$$\sigma_M : kE^1 \otimes_{kE^0} M \to M \otimes_{kF^0} kF^1 \qquad \sigma_N : kF^1 \otimes_{kF^0} N \to N \otimes_{kE^0} kE^1$$

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such that these two diagrams commute:

$$\begin{array}{ccc} kE^1 \otimes_{kE^0} (M \otimes_{kF^0} N) & \xrightarrow{\sigma_M \# \sigma_N} & (M \otimes_{kF^0} N) \otimes_{kE^0} kE^1 \\ \downarrow{\scriptstyle \mathrm{id} \otimes \omega_E} & & \downarrow{\scriptstyle \omega_E \otimes \mathrm{id}} \\ kE^1 \otimes_{kE^0} (kE^1)^{\otimes n} & \xrightarrow{\nu_n^E} & (kE^1)^{\otimes n} \otimes_{kE^0} kE^1 \end{array}$$

$$\begin{array}{ccc} kF^1 \otimes_{kF^0} (N \otimes_{kE^0} M) & \xrightarrow{\sigma_N \# \sigma_M} & (N \otimes_{kE^0} M) \otimes_{kF^0} kF^1 \\ \downarrow{\scriptstyle \mathrm{id} \otimes \omega_F} & & \downarrow{\scriptstyle \omega_F \otimes \mathrm{id}} \\ kF^1 \otimes_{kF^0} (kF^1)^{\otimes n} & \xrightarrow{\nu_n^F} & (kF^1)^{\otimes n} \otimes_{kF^0} kF^1 \end{array}$$

Condition (Com)

Proposition. For finite graphs with no sinks,

Strong Shift Equivalence

$\Longrightarrow$ Module Shift Equivalence + (Com)

$\Longrightarrow$ Shift Equivalence

Proof of first implication: Associativity.

Condition (Com)

Is Strong Shift Equivalence decidable?

This question has been open for awhile.

Here's one possible (new) line of attack.

Step 1: Show that Module Shift Equivalence + (Com) is decidable (or not)

Step 2: Show that Module Shift Equivalence + (Com) is equivalent to Strong Shift Equivalence.

Leavitt path algebras, and Hazrat's Conjecture

Let E be a finite directed graph, and k any field.

Build the *Leavitt path algebra* $L_k(E)$ *of* E *with coefficients in* k.

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Let E and F be finite graphs. **General question**:

When are $L_k(E)$ and $L_k(F)$ "the same" ??

Rephrased: can we find "easily" computable quantities associated to E and F (and maybe k) so that "sameness" of the quantities is equivalent to "sameness" of the Leavitt path algebras?

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Are you kidding me ?!?!?

Leavitt path algebras, and Hazrat's Conjecture

What does $L_k(E)$ and $L_k(F)$ being "the same" mean?

There are many natural possibilities.

Leavitt path algebras, and Hazrat's Conjecture

What does $L_k(E)$ and $L_k(F)$ being "the same" mean?
There are many natural possibilities.

$L_k(E)$ is isomorphic to $L_k(F)$ (as rings, or as k-algebras); or

$L_k(E)$ is Morita equivalent to $L_k(F)$; or

$L_k(E)$ is graded isomorphic to $L_k(F)$ (natural $\mathbb{Z}$-grading); or

$L_k(E)$ is graded Morita equivalent to $L_k(F)$ (natural $\mathbb{Z}$-grading).

Leavitt path algebras, and Hazrat's Conjecture

For the two graded notions of "sameness":

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there is **Hazrat's Conjecture**.

(Focus today on graded Morita equivalence.)

Note: the current approaches to all four questions involve an analysis of the finitely generated projective modules over the corresponding Leavitt path algebras.

A brief review, and some background

$\mathbb{Z}$-graded rings.

A ring R is $\mathbb{Z}$-graded in case:

1. $(R, +) = \oplus_{t \in \mathbb{Z}} R_t$ as abelian groups, and
2. $R_t \cdot R_u \subseteq R_{t+u}$ for all $t, u \in \mathbb{Z}$.

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Familiar examples: $k[x]$, $k[x, x^{-1}]$.

Graded modules, the category $\mathrm{Gr}\text{-}R$

Graded modules and graded homomorphisms.

R a graded ring, M_R a right R-module.

M is *graded* in case:

$$M = \oplus_{t \in \mathbb{Z}} M_t, \quad \text{and} \quad M_u R_t \subseteq M_{t+u} \text{ for all } t, u \in \mathbb{Z}.$$

If M, N are graded right R-modules, an R-homomorphism $f : M \to N$ is called *graded* in case $f(M_t) \subseteq N_t$ for all $t \in \mathbb{Z}$.

$$\mathrm{Gr}\text{-}R$$

denotes the category of graded right R-modules with graded homomorphisms.

Graded modules: some terminology

For a graded right A-module M and $i \in \mathbb{Z}$,

the *i-suspension of M*,

denoted $M(i)$, is the graded right A-module having $M(i) = M$, with grading given by

$$M(i)_j = M_{i+j}.$$

In other words, the grading on $M(i)$ is gotten by "shifting" the grading on M by i units.

Graded modules: some terminology

For $i \in \mathbb{Z}$, $\mathcal{T}_i$ denotes the *i-suspension functor* (or *i-shift functor*)

$$\mathcal{T}_i : \mathrm{Gr}\text{-}A \to \mathrm{Gr}\text{-}A$$

given by $M \mapsto M(i)$ on objects, and the identity on morphisms.

A functor $\Phi : \mathrm{Gr}\text{-}A \to \mathrm{Gr}\text{-}B$ is called *graded* when

$$\Phi \circ \mathcal{T}_\alpha = \mathcal{T}_\alpha \circ \Phi$$

for each $\alpha \in \mathbb{Z}$.

Graded modules: some terminology

A graded functor $\Phi : \mathrm{Gr}\text{-}A \to \mathrm{Gr}\text{-}B$ is a *graded equivalence* if there is a graded functor $\Psi : \mathrm{Gr}\text{-}B \to \mathrm{Gr}\text{-}A$ such that Φ and Ψ compose appropriately to the identity functors on each category.

If there is a graded equivalence $\mathrm{Gr}\text{-}A \ \to \ \mathrm{Gr}\text{-}B$, we say

A and *B* are *graded equivalent*

or, more formally, *graded Morita equivalent*.

Graded modules: some terminology

For any graded ring A, we let U_A (or simply by U) denote the *forgetful functor*

$$U_A : \mathrm{Gr}\text{-}A \to \mathrm{Mod}\text{-}A.$$

A functor $\Phi' : \mathrm{Mod}\text{-}A \to \mathrm{Mod}\text{-}B$ is called a *graded functor* if there is a graded functor $\Phi : \mathrm{Gr}\text{-}A \to \mathrm{Gr}\text{-}B$ such that

$$U_B \circ \Phi = \Phi' \circ U_A$$

as functors from $\mathrm{Gr}\text{-}A$ to $\mathrm{Mod}\text{-}B$. In this situation the functor Φ is called an *associated graded functor* of Φ'.

Graded modules: some terminology (it's less-than-optimal)

A functor $\Phi' : \mathrm{Mod}\text{-}A \to \mathrm{Mod}\text{-}B$ is called a *graded equivalence* if it is both graded and an equivalence. If such exists, we say that

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Graded modules: some terminology (it's less-than-optimal)

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A and B are *graded equivalent*

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(Notation is seemingly not optimal) But ....

Theorem: Let A and B be unital graded rings. Then

there is a graded equivalence $\Phi : \text{Gr-}A \to \text{Gr-}B$

if and only if

there is a graded equivalence $\Delta : \text{Mod-}A \to \text{Mod-}B$.

Theorem 2.3.8 in R. Hazrat, Graded rings and graded Grothendieck groups. London Mathematical Society Lecture Note Series, **435**. Cambridge University Press, Cambridge, 2016. vii+235 pp.

Graded equivalences

Big example: (The only one we'll need) Suppose ${}_AX_B$ is a graded bimodule. Then

$$F : \text{Mod-}A \to \text{Mod-}B \quad \text{via} \quad F(M_A) = M_A \otimes_A {}_AX_B$$

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Graded equivalences

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$$F : \text{Mod-}A \to \text{Mod-}B \quad \text{via} \quad F(M_A) = M_A \otimes_A {}_AX_B$$

is a graded functor. Consequently ...

Suppose ${}_AX_B$ and ${}_BY_A$ are graded bimodules, and suppose that

$${}_AX_B \otimes_B {}_BY_A \cong {}_AA_A \quad \text{and} \quad {}_BY_A \otimes_A {}_AX_B \cong {}_BB_B$$

as (not-necessarily-graded) bimodules. Then A is graded equivalent to B.

Leavitt path algebras

Any Leavitt path algebra is $\mathbb{Z}$-graded, with grading given by setting

$$pq^* \in L_k(E)_n \ \text{ in case } \ \ell(p) - \ell(q) = n$$

for paths p, q in E, and $n \in \mathbb{Z}$.

Hazrat's Conjecture

Hazrat's Conjecture: Let E and F be finite essential graphs, and k any field. Then

- the Leavitt path algebras $L_k(E)$ and $L_k(F)$ are graded equivalent

if and only if

- the matrices A_E and A_F are shift equivalent.

($\Longrightarrow$) is known; ($\Longleftarrow$) is still unresolved

Hazrat's Conjecture

But this is known:

Theorem (Hazrat, 2013): Suppose E and F are finite essential graphs. If A_E and A_F are *strong* shift equivalent, then the Leavitt path algebras $L_k(E)$ and $L_k(F)$ are graded equivalent.

Hazrat's Conjecture: our contribution

Here's an outline of what we've done in our current work.

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Step 1. Show that shift equivalence of adjacency matrices can equivalently be recast as Module Shift Equivalence.

Step 2. Focusing on one of these four isomorphisms, define a left $L_k(E)$-module structure on a specially-built $kE^0 - L_k(F)$ bimodule X. X thereby becomes a (graded) $L_k(E) - L_k(F)$ bimodule. This is *not* expected.

"Bridging bimodule"

Hazrat's Conjecture: our contribution

Step 3. Module Shift Equivalence allows for an extra, "natural" condition to be appended. "Condition (Com)"

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Step 4. Show that Condition (Com), when added to Module Shift Equivalence of adjacency matrices, gives that tensoring by X induces a graded equivalence from $L_k(E)$-Mod to $L_k(F)$-Mod.

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Step 3. Module Shift Equivalence allows for an extra, "natural" condition to be appended. "Condition (Com)"

Step 4. Show that Condition (Com), when added to Module Shift Equivalence of adjacency matrices, gives that tensoring by X induces a graded equivalence from $L_k(E)$-Mod to $L_k(F)$-Mod.

Step 5. **Conclusion**: Hazrat's Conjecture holds in the situation where we modify the hypotheses (shift equivalence of adjacency matrices) by adding Condition (Com).

Step 4: (MSE) + (Com) $\Rightarrow$ graded Morita equivalence

Main Theorem: Let k be any field, and let E and F be finite graphs with no sinks. Suppose there exist M, N, n, and bimodule isomorphisms

$$\omega_E : M \otimes_{kF^0} N \to (kE^1)^{\otimes n} \qquad \omega_F : N \otimes_{kE^0} M \to (kF^1)^{\otimes n}$$

$$\sigma_M : kE^1 \otimes_{kE^0} M \to M \otimes_{kF^0} kF^1 \qquad \sigma_N : kF^1 \otimes_{kF^0} N \to N \otimes_{kE^0} kE^1$$

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satisfying Condition (Com). Then $L_k(E)$ and $L_k(F)$ are graded Morita equivalent.

Proof. Build the $L_k(E) - L_k(F)$ bridging bimodule X, using $\sigma_M : kE^1 \otimes_{kE^0} M \to M \otimes_{kF^0} kF^1$. Build the $L_k(F) - L_k(E)$ bridging bimodule Y, using $\sigma_N : kF^1 \otimes_{kF^0} N \to N \otimes_{kE^0} kE^1$. Show that (Com) implies that $X \otimes_{L_k(F)} Y \cong L_k(E)$ and $Y \otimes_{L_k(E)} X \cong L_k(F)$ as bimodules. $\square$

5. Conclusion: Hazrat's Conjecture, maybe?

Main Theorem, rephrased: Hazrat's Conjecture holds, with

the hypothesis "shift equivalence"

replaced by

the hypothesis "module shift equivalence plus (Com)".

5. Conclusion: Hazrat's Conjecture, maybe?

Corollary of Main Theorem: Let E and F be finite graphs with no sinks. If A_E is strong shift equivalent to A_F then $L_k(E)$ is graded Morita equivalent to $L_k(F)$.

(Remark: this slightly improves Hazrat's previous result, it removes the "no sources" condition.)

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For finite directed graphs with no sinks,

Strong Shift Equivalence (SSE)

$\Longrightarrow$ (MSE) + (Com)

$\Longrightarrow$ Shift Equivalence (SE).

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Strong Shift Equivalence (SSE)

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And we have shown that (MSE) + (Com) implies graded Morita equivalence.

But remember, there is almost no room between (SSE) and (SE) !!!

Thank you
for your time.

Hazrat's Conjecture

One of the two most-discussed currently-open questions in the subject of Leavitt path algebras is

"Hazrat's Conjecture"

Let E and F be finite essential graphs.

Suppose there is an "order-preserving" $\mathbb{Z}[x, x^{-1}]$-module isomorphism between $K_0^{gr}(L_k(E))$ and $K_0^{gr}(L_k(F))$.

Then $L_k(E)$ and $L_k(F)$ graded Morita equivalent.

Hazrat's Conjecture, and connections to previous ideas

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Theorem: Let E and F be finite essential graphs, and k any field. The following are equivalent.

1. There is an order-preserving $\mathbb{Z}[x, x^{-1}]$-module isomorphism between $K_0^{gr}(L_k(E))$ and $K_0^{gr}(L_k(F))$.
2. The adjacency matrices A_E and A_F are shift equivalent.

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Idea of Proof: Shift equivalence of adjacency matrices A_E and A_F is equivalent to an isomorphism between the "Krieger dimension triples" of E and F (done in Lind and Marcus); and these triples turn out to give the right info about the graded K_0-groups as $\mathbb{Z}[x, x^{-1}]$-modules.

Graded finitely generated projective modules

If R is graded then

$$\mathcal{V}^{gr}(R)$$

denotes the graded-isomorphism classes of graded finitely generated projective right R-modules.

$\mathcal{V}^{gr}(R)$ is an abelian monoid under $\oplus$.

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$\mathcal{V}^{gr}(R)$ is an abelian monoid under $\oplus$.

And there is a module action of $\mathbb{Z}[x, x^{-1}]$ on $\mathcal{V}^{gr}(R)$, via the suspension functor.

And in turn this induces a $\mathbb{Z}[x, x^{-1}]$-module action on $K_0^{gr}(R)$.

Step 1: Polymorphisms

Definition: A *polymorphism* from a set V to a set W is a 5-tuple $E := (V, W, E^1, r, s)$ consisting of sets V, W, and E^1 with functions $s : E^1 \to V$ and $r : E^1 \to W$.

Example: a directed graph, $V = W$.

Step 1: Polymorphisms

Definition: For fixed V and W, two polymorphisms

$$E := (V, W, E^1, r_E, s_E) \quad \text{and} \quad G := (V, W, G^1, r_F, s_F)$$

are *isomorphic* when there exists a bijection $\phi : E^1 \to G^1$ with

$$s_G(\phi(e)) = \phi(s_E(e)) \ \text{ and } \ r_G(\phi(e)) = \phi(r_E(e))$$

for all $e \in E^1$.

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for all $e \in E^1$.

Intuitively, an isomorphism from E to G is just a relabelling of the edge sets from each v to each w.

Step 1: Polymorphisms

Definition: For polymorphism $E := (V, W, E^1, r, s)$, the adjacency matrix A_E of E is the $V \times W$ matrix

$$A_E(v, w) := \#\{e \in E^1 : s(e) = v \text{ and } r(e) = w\}.$$

So $A_E \in \mathrm{M}_{V\times W}(\mathbb{Z}_{\geq 0})$. And any $M \in \mathrm{M}_{V\times W}(\mathbb{Z}_{\geq 0})$ produces a polymorphism from V to W.

Clearly if E and G are polymorphisms from V to W, then E is isomorphic to G if and only if $A_E = A_G$.

Step 1: Polymorphisms

Definition: Let k be a field. For a set S,

$$kS := k\text{-vector space with basis } S; \quad \text{i.e., } \oplus_{s \in S} k.$$

Let $E := (V, W, E^1, r, s)$ be a polymorphism from V to W.

kV and kW are naturally k-algebras,
isomorphic to $\oplus_{v \in V} k$ and $\oplus_{w \in W} k$, resp.

kE^1 has no 'natural' ' algebra structure.
But kE^1 is a $kV - kW$-bimodule in the natural way.

Step 1: Polymorphisms

When $E = (E^0, E^1, r, s)$ is a graph, then kE^1 is a kE^0–kE^0-bimodule.

Also, kE^1 embeds into $L_k(E)$, so we can view kE^1 as a kE^0-submodule of $L_k(E)$.

Step 1: Polymorphisms

Theorem: [Graph Shift Equiv. $\Leftrightarrow$ Module Shift Equiv.] Let k be a field, and let E and F be finite graphs with no sinks. Then

A_E and A_F are shift equivalent (with exponent n) if and only if:

(MSE) There exists a kE^0–kF^0-bimodule M, a kF^0–kE^0-bimodule N, and a positive integer n for which there are bimodule isomorphisms

$$(kE^1)^{\otimes n} \cong M \otimes_{kF^0} N \qquad (kF^1)^{\otimes n} \cong N \otimes_{kE^0} M$$
$$kE^1 \otimes_{kE^0} M \cong M \otimes_{kF^0} kF^1 \qquad kF^1 \otimes_{kF^0} N \cong N \otimes_{kE^0} kE^1.$$

Proof: Not bad; uses properties of polymorphisms.

Step 2: Module shift equivalence; bridging bimodule

The theorem allows us to replace Condition (GSE) in Hazrat's Conjecture with Condition (MSE).

Step 2: Module shift equivalence; bridging bimodule

The theorem allows us to replace Condition (GSE) in Hazrat's Conjecture with Condition (MSE).

So what?

Condition (GSE) asserts that certain matrix products are equal,

while

Condition (MSE) requires certain bimodules to be isomorphic.

We will eventually add an extra hypothesis to Condition (MSE) in a very useful way.

Step 2: Module shift equivalence; bridging bimodule

First ...

Definition. Let E and F be finite graphs with no sinks. Suppose M is a $kE^0 - kF^0$-bimodule for which there is a $kE^0 - kF^0$ bimodule isomorphism

$$kE^1 \otimes_{kE^0} M \cong M \otimes_{kF^0} kF^1 .$$

(N.b.: this is one of the four bimodule isomorphism conditions that appear in the theorem relating GSE to MSE.)

Choose a specific isomorphism $\sigma : kE^1 \otimes_{kE^0} M \to M \otimes_{kF^0} kF^1$.

(M, σ) is called a *specified conjugacy pair from E to F*.

Step 2: Module shift equivalence; bridging bimodule

Theorem: Let E and F be finite graphs with no sinks. Let (M, σ) be a specified conjugacy pair from E to F.
Then the $kE^0 - L_k(F)$-bimodule

$$M \otimes_{kF^0} L_k(F)$$

admits a left $L_k(E)$-action which makes $M \otimes_{kF^0} L_k(F)$ into a graded $L_k(E)$–$L_k(F)$-bimodule.

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Proof: Uses an idea similar to one used in C*-algebra setting by Eryüzlü. Use $\sigma : kE^1 \otimes_{kE^0} M \to M \otimes_{kF^0} kF^1$ to define actions

$$v \cdot (m \otimes S), \quad e \cdot (m \otimes S), \quad \text{and} \quad e^* \cdot (m \otimes S)$$

for $m \in M$ and $S \in L_k(F)$, and show that these satisfy the appropriate Leavitt path algebra relations in $L_k(E)$.

Step 3: Condition (Com)

Reminder: (MSE) is: There exists a kE^0–kF^0-bimodule M, a kF^0–kE^0-bimodule N, and a positive integer n for which

$$(kE^1)^{\otimes n} \cong M \otimes_{kF^0} N \qquad (kF^1)^{\otimes n} \cong N \otimes_{kE^0} M$$
$$kE^1 \otimes_{kE^0} M \cong M \otimes_{kF^0} kF^1 \qquad kF^1 \otimes_{kF^0} N \cong N \otimes_{kE^0} kE^1,$$

where the isomorphisms are bimodule isomorphisms.

Now suppose there exists some collection of such bimodule isomorphisms that behaves nicely.

Condition (Com)

Step 3: Condition (Com)

Specifically, suppose $\exists$ M, N, n, and bimodule isomorphisms

$$\omega_E : M \otimes_{kF^0} N \to (kE^1)^{\otimes n} \qquad \omega_F : N \otimes_{kE^0} M \to (kF^1)^{\otimes n}$$

$$\sigma_M : kE^1 \otimes_{kE^0} M \to M \otimes_{kF^0} kF^1 \qquad \sigma_N : kF^1 \otimes_{kF^0} N \to N \otimes_{kE^0} kE^1$$

such that these two diagrams commute:

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such that these two diagrams commute:

$$\begin{array}{ccc} kE^1 \otimes_{kE^0} (M \otimes_{kF^0} N) & \xrightarrow{\sigma_M \# \sigma_N} & (M \otimes_{kF^0} N) \otimes_{kE^0} kE^1 \\ \downarrow {\scriptstyle \mathrm{id} \otimes \omega_E} & & \downarrow {\scriptstyle \omega_E \otimes \mathrm{id}} \\ kE^1 \otimes_{kE^0} (kE^1)^{\otimes n} & \xrightarrow{\nu_n^E} & (kE^1)^{\otimes n} \otimes_{kE^0} kE^1 \end{array}$$

$$\begin{array}{ccc} kF^1 \otimes_{kF^0} (N \otimes_{kE^0} M) & \xrightarrow{\sigma_N \# \sigma_M} & (N \otimes_{kE^0} M) \otimes_{kF^0} kF^1 \\ \downarrow {\scriptstyle \mathrm{id} \otimes \omega_F} & & \downarrow {\scriptstyle \omega_F \otimes \mathrm{id}} \\ kF^1 \otimes_{kF^0} (kF^1)^{\otimes n} & \xrightarrow{\nu_n^F} & (kF^1)^{\otimes n} \otimes_{kF^0} kF^1 \end{array}$$